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Cardinal Exponential Splines and Laplace Transform

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1. INTRODUCTION

It is the aim of a series of preceding papers [5–9] to treat some representation and convergence problems arising in the theory of cardinal logarithmic spline functions by real and complex integral transform methods. In particular, the Laplace and the inverse Mellin transform turned out to be pliable and versatile tools for these purposes. It is the aim of the present note to deal with cardinal exponential splines (in the sense of Schoenberg [11, 12]) by means of an integral representation formula for the basis splines that has been already mentioned in [9]. The formula will be established in Theorem 1 via the inverse Laplace transform. As a consequence, it implies a contour integral representation for the cardinal exponential splines (Theorem 3) which is the central result of this note. The line integral involved can be evaluated by an application of the calculus of residues (Theorem 4). In this way, for instance, the pointwise convergence of the cardinal exponential spline interpolants on the whole real line $\mathbb{R}$ towards the exponential function when their degree tends to infinity will be established without any difficulty (Theorem 6). Thus, the asymptotic behaviour of the cardinal exponential spline interpolants and the cardinal logarithmic splines turns out to be totally different. In both cases, however, the contour integral representation method exhibits as an effective technique.

2. TRUNCATED POWER FUNCTIONS

Let $m \geqslant 1$ be a fixed natural number and suppose that $z \in \mathbb{C}$ belongs to the complex open right half-plane $\operatorname{Re} z > 0$. By a change of variable the identity

$$\int_{\mathbb{R}_+} e^{-zx} x^m \, dx = \frac{1}{z^{m+1}} \Gamma(m+1) = \frac{m!}{z^{m+1}}$$

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obtains, i.e., the one-sided Laplace transform of the monomial function $x \rightsquigarrow x^m$ at the point $z \in \mathbb{C}$ equals to $m!/z^{m+1}$ provided that $\operatorname{Re} z > 0$. If we denote, as usually, for each mapping $f: \mathbb{R} \to \mathbb{R}$ by $f_+ = \sup(f, 0)$ its positive part, an application of the inversion theorem for the bilateral Laplace transform furnishes the integral representation formula for the truncated power functions

$$x_+^m = (x_+)^m = \frac{m!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{xz}}{z^{m+1}} \, dz \qquad (c > 0) \tag{1}$$

which is valid for all $x \in \mathbb{R}$. It should be observed that in (1) the integral along a straight line in the open right half-plane parallel to the imaginary axis is independent of the particular choice of $c > 0$. From (1) one also concludes that the integral representation formula

$$(-x)_+^m = ((-x)_+)^m = \frac{(-1)^{m+1} m!}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{xz}}{z^{m+1}} \, dz \qquad (d < 0) \tag{2}$$

holds for all $x \in \mathbb{R}$ and that the line integral occurring in (2) is independent of the choice of the constant $d < 0$ again.

It should be emphasized that the fundamental integral representation formulae (1) and (2) of the truncated power functions $\mathbb{R} \ni x \rightsquigarrow x_+^m \in \mathbb{R}_+$, $R \ni x \rightsquigarrow (-x)_+^m \in \mathbb{R}_+$ $(m \in \mathbb{N}^\times)$ along vertical lines in the open right, resp. left, half-plane form an essential ingredient of our approach to the cardinal exponential splines. For another kind of integral representation formula, see Section 8.

3. Basis Splines

For each $m \in \mathbb{N}^\times$ let $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ denote the vector space over the field $\mathbb{C}$ of all complex cardinal spline functions of degree m on $\mathbb{R}$ having (equidistant) knots at the integer points $n \in \mathbb{Z}$ on the real line $\mathbb{R}$, i.e., the vector space of all complex-valued functions $s \in \mathscr{C}^{m-1}(\mathbb{R})$ such that the restriction of s to each compact subinterval $[n, n+1]$ $(n \in \mathbb{Z})$ of $\mathbb{R}$ is a polynomial of degree $\leqslant m$ with complex coefficients.

Obviously, for each $k \in \mathbb{Z}$, the truncated power function

$$\mathbb{R} \ni x \rightsquigarrow (x-k)_+^m \tag{3}$$

belongs to the space $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$. In particular, the so-called forward basis spline of degree m given by the linear combinations of the splines (3) and their reflections

$$b_m : \mathbb{R} \ni x \rightsquigarrow \frac{1}{m!} \sum_{0 \leqslant k \leqslant m+1} (-1)^k \binom{m+1}{k} (x-k)_+^m, \tag{4}$$

$$b_m : \mathbb{R} \ni x \rightsquigarrow \frac{(-1)^{m+1}}{m!} \sum_{0 \leqslant k \leqslant m+1} (-1)^k \binom{m+1}{k} (k-x)_+^m \tag{4'}$$

is an element of $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ and satisfies the conditions

$$\operatorname{Supp}(b_m) \subseteq [0, m+1], \qquad \int_0^{m+1} b_m(t)\, dt = 1. \tag{5}$$

Conversely, any function $b_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ that satisfies the conditions (5) is given by (4) or (4′). In view of (4), (4′) and (1), (2) we obtain the following result:

THEOREM 1. *For each $m \in \mathbb{N}^\times$ the basis spline $b_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree m admits the integral representations*

$$b_m(x) = \sum_{0 \leqslant k \leqslant m+1} (-1)^k \binom{m+1}{k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{(x-k)z}}{z^{m+1}}\, dz \qquad (c > 0), \tag{6}$$

$$b_m(x) = \sum_{0 \leqslant k \leqslant m+1} (-1)^k \binom{m+1}{k} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{(x-k)z}}{z^{m+1}}\, dz \qquad (d < 0), \tag{6'}$$

which are valid for all $x \in \mathbb{R}$. The line integrals occurring in (6) and (6′) are independent of the particular choices of the real constants $c > 0$ and $d < 0$, respectively.

COROLLARY. *For all $x \in \mathbb{R}$ the homogeneous linear difference equation*

$$b_m(x) - b_m(m+1-x) = 0 \qquad (m \in \mathbb{N}^\times) \tag{7}$$

holds.

The importance of the basis spline b_m lies in the fact that its translates within the grid $\mathbb{Z}$ form a base of the complex vector space $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$. More precisely, for any cardinal spline function $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ there exists a unique bi-infinite sequence $(C_n)_{n \in \mathbb{Z}}$ of complex numbers such that the representation formula

$$s_m = \sum_{n \in \mathbb{Z}} C_n b_m(\cdot - n) \tag{8}$$

holds. In view of the "small support" condition pointed out in (5) there occurs in (8) only a finite number of summands $\neq 0$, i.e., there is no convergence problem with respect to the bi-infinite series (8).

For an account of basis splines the reader is referred to the fundamental paper by Curry and Schoenberg [1] and, for instance, to the recent book by De Boor [2] which emphasizes the computational aspects of this notion.

4. Cardinal Exponential Splines and Their Contour Integral Representations

Let $h \neq 0$ denote a fixed complex number and consider the homogeneous linear difference equation of the first order with constant coefficients

$$f(x+1) - hf(x) = 0 \qquad (x \in \mathbb{R}). \tag{9}$$

It follows from (8) that $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ is a solution of (9) if and only if the coefficients $(C_n)_{n \in \mathbb{Z}}$ of s_m admit the form $C_n = C_0 \cdot h^n$ for all $n \in \mathbb{Z}$, i.e., if and only if s_m may be represented by the formula

$$s_m = C_0 \cdot \sum_{n \in \mathbb{Z}} h^n b_m(\cdot - n), \tag{10}$$

where $C_0 \in \mathbb{C}$ denotes an arbitrary constant. Following the terminology of Schoenberg [11, 12], in this case $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ is called a cardinal exponential spline of degree m and weight h. In this connection also see Greville *et al.* [3].

A short computation based on Theorem 1 and the binomial theorem furnish

THEOREM 2. *For each number $m \in \mathbb{N}^\times$ the cardinal exponential splines $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree m and complex weight $h \neq 0$ admit the representation formulae*

$$s_m(x) = C_0 \left(1 - \frac{1}{h}\right)^{m+1} \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \int_{c - i\infty}^{c + i\infty} (he^{-z})^n \frac{e^{xz}}{z^{m+1}}\, dz \qquad (c > 0), \tag{11}$$

$$s_m(x) = C_0 \left(1 - \frac{1}{h}\right)^{m+1} \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \int_{d - i\infty}^{d + i\infty} (he^{-z})^n \frac{e^{xz}}{z^{m+1}}\, dz \qquad (d < 0) \tag{11'}$$

for all $x \in \mathbb{R}$. In (11) and (11′), $C_0 \in \mathbb{C}$ denotes an arbitrary constant.

Suppose that $h \in \mathbb{C}^\times$ does not belong to the unit circle $U = \{w \in \mathbb{C} \wr |w| = 1\}$. In the case when $|h| > 1$, choose real numbers c, c' such that

$$0 < c' < \log |h| < c$$

holds. In the other case when $0 < |h| < 1$ assume that the real numbers c, c' satisfy the conditions

$$c' < \log |h| < c < 0.$$

In any case, introduce the vertical lines

$$L = \{z \in \mathbb{C} \wr \operatorname{Re} z = c\}, \qquad L' = \{z \in \mathbb{C} \wr \operatorname{Re} z = c'\}.$$

Then $L \cup L'$ forms the boundary of a closed vertical strip in the open right, resp. left, half-plane with the compact basis $[c', c]$ on the real axis. Proceeding as in [9], let L and L' be equipped with a positive orientation such that their juxtaposition

$$L_0 = L \vee L' \tag{12}$$

forms a circuit in the one-point compactification of $\mathbb{C}$ having the topological index $\operatorname{Ind}_{L_0}(\log |h|) = 1$ with respect to the point $\log |h|$ on the real axis of $\mathbb{C}$. Then let the bi-infinite series (11) resp. (11′) be decomposed into two parts such that the first one includes the summation over all numbers $n \in \mathbb{N}$ and the corresponding line integrals are along L whereas the second part is concerned with the summation over all integers $n \leqslant -1$ and the corresponding line integrals are along L'. In view of the uniform convergence of both sums with respect to the variable z it is permissible to interchange the order of integration and summation in both cases. If we put the two parts together after the break and if we introduce for all triplets $(m, h, x) \in \mathbb{N}^\times \times (\mathbb{C}^\times - U) \times \mathbb{R}$ the meromorphic function

$$F_{m,h,x} : z \rightsquigarrow \frac{e^{(x+1)z}}{(e^z - h)\, z^{m+1}}, \tag{13}$$

then by Theorem 2 supra the central result of this note is established.

THEOREM 3. *For the cardinal exponential splines $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree $m \in \mathbb{N}^\times$ and weight $h \in \mathbb{C}^\times - U$ the contour integral representation formula*

$$s_m(x) = C_0 \left(1 - \frac{1}{h}\right)^{m+1} \frac{1}{2\pi i} \int_{L_0} F_{m,h,x}(z)\, dz \qquad (x \in \mathbb{R}) \tag{14}$$

holds. The kernel $F_{m,h,x}$ is defined by (13), *the contour L_0 is given according to* (12) *and $C_0 \in \mathbb{C}$ denotes an arbitrary constant.*

The line integral occurring in (14) along the circuit L_0 can be evaluated by an application of the calculus of residues. For this aim we observe that the meromorphic function (13) admits a pole of order $m+1$ at the origin of $\mathbb{C}$ and simple poles at the zeros $(z_k(h))_{k\in\mathbb{Z}}$ of the function $z \rightsquigarrow e^z - h$. Since we have

$$\mathrm{Ind}_{L_0}(0) = 0, \qquad \mathrm{Ind}_{L_0}(z_k(h)) = 1 \qquad (k \in \mathbb{Z})$$

an application of Cauchy's residue theorem yields the identity

$$s_m(x) = C_0 \left(1 - \frac{1}{h}\right)^{m+1} \sum_{k\in\mathbb{Z}} \mathrm{Res}(F_{m,h,x}, z_k(h)) \tag{15}$$

for all $x \in \mathbb{R}$. Note that $z_k(h) \neq 0$ for all $k \in \mathbb{Z}$ and that

$$\mathrm{Res}(F_{m,h,x}, z_k(h)) = \frac{e^{xz_k(h)}}{z_k^{m+1}(h)}$$

holds for all triplets $(m, h, x) \in \mathbb{N}^\times \times (\mathbb{C}^\times - U) \times \mathbb{R}$. Since the bi-infinite sequence $(z_k(h))_{k\in\mathbb{Z}}$ of equidistant poles is located on the straight line $\{w \in \mathbb{C} \mid \mathrm{Re}\, w = \log|h|\}$ parallel to the imaginary axis of $\mathbb{C}$, Fig. 1 turns out in the case $|h| > 1$ (and similar in the case $|h| < 1$).

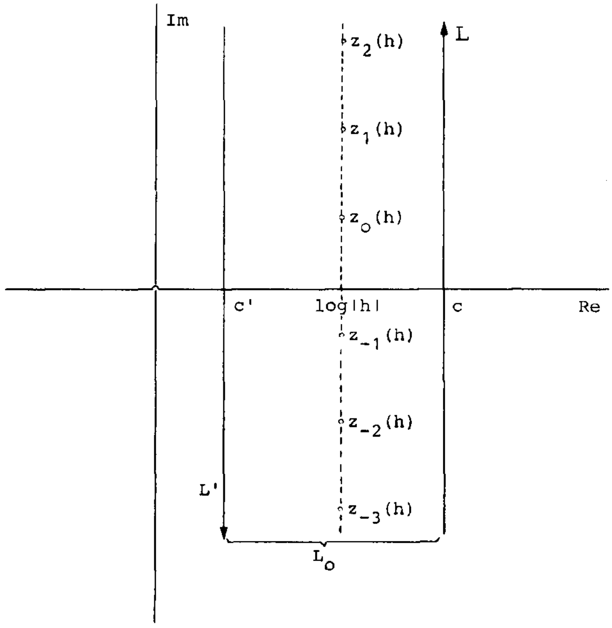

FIGURE 1

As a consequence of Theorem 3, the identity (15) establishes

THEOREM 4. *Let the numbers $m \in \mathbb{N}^\times$ and $h \in \mathbb{C}^\times - U$ be fixed. Then the cardinal exponential splines $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree m and weight h admit the form*

$$s_m(x) = C_0 \left(1 - \frac{1}{h}\right)^{m+1} \sum_{k \in \mathbb{Z}} \frac{e^{xz_k(h)}}{z_k^{m+1}(h)} \qquad (x \in \mathbb{R}), \tag{16}$$

where $C_0 \in \mathbb{C}$ denotes an arbitrary constant.

5. CARDINAL EXPONENTIAL INTERPOLANTS

Taking into account the homogeneous linear difference equation of the first order (9) that is satisfied by the cardinal exponential splines s_m of degree $m \in \mathbb{N}^\times$ and weight $h \neq 0$ the normalization condition

$$s_m(0) = 1 \tag{17}$$

exhibits to be necessary and sufficient that s_m verifies the interpolation condition

$$s_m(n) = h^n \qquad (n \in \mathbb{Z})$$

on the grid $\mathbb{Z}$. In this case, $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ is called a cardinal exponential spline interpolant of degree m with respect to the bi-infinite geometric sequence $(h^n)_{n \in \mathbb{Z}}$.

THEOREM 5. *Let the numbers $m \in \mathbb{N}^\times$ and $h \in \mathbb{C}^\times - U$ be given. There exists one and only one cardinal exponential spline interpolant s_m of degree m with respect to $(h^n)_{n \in \mathbb{Z}}$ if and only if the condition*

$$q_{m+1}(h) = \sum_{k \in \mathbb{Z}} \frac{1}{z_k^{m+k}(h)} \neq 0 \tag{18}$$

holds. In this case s_m admits the form

$$s_m : \mathbb{R} \ni x \rightsquigarrow \frac{1}{q_{m+1}(h)} \sum_{k \in \mathbb{Z}} \frac{e^{xz_k(h)}}{z_k^{m+1}(h)} \qquad (m \in \mathbb{N}^\times). \tag{19}$$

6. THE FUNCTIONS $(q_{m+1})_{m\in\mathbb{N}^\times}$

In the case when $C_0 \neq 0$ by comparing the identities (10) and (16) and using (5) and (7) we obtain

$$q_{m+1}(h) = \frac{1}{(h-1)^{m+1}} \sum_{1\leqslant n\leqslant m} b_m(n)\, h^n = \frac{1}{(h-1)^{m+1}}\, p_m(h). \qquad (m\in\mathbb{N}^\times) \tag{20}$$

Therefore the functions q_{m+1} do not possess more than $m-1$ zeros $\neq 0$ in the open subset $\mathbb{C}-\{1\}$ of $\mathbb{C}$. Let $T_\alpha = \mathbb{R}_+ e^{i\alpha}$ ($\alpha \in\,]-\pi, +\pi]$) denote an arbitrary closed half-line in the complex plane $\mathbb{C}$ that starts from the origin of $\mathbb{C}$ in the direction α. Choose a holomorphic logarithm in the complex plane cut along the ray T_α, i.e., in the open set $\mathbb{C}-T_\alpha$ (cf. Fig. 2 below). Then it follows from (18) that the derivative of q_{m+1} satisfies

$$q'_{m+1}(h) = -\frac{m+1}{h}\, q_{m+2}(h) \tag{21}$$

for all numbers $h \in \mathbb{C}-(U\cup T_\alpha)$. In view of (20) the identity (21) can be extended to all numbers $h\in\mathbb{C}-\{1\}$. Let h_0 denote the lowest zero of the polynomial p_m of degree m with real coefficients located on the closed left real half-line $\mathbb{R}_- = \{r\in\mathbb{R} \mid r \leqslant 0\}$. Then we obtain by combining (20) and (21) the identity

$$p_{m+1}(h_0) = -\frac{1}{m+1}\, h_0(h_0-1)\, p'_m(h_0) \tag{22}$$

for all $m\in\mathbb{N}^\times$. Consequently, the set of all zeros of the functions $(q_{m+1})_{m\in\mathbb{N}^\times}$ is located on the closed half-line $\mathbb{R}_-$. Thus, by Theorem 5 supra, there exists for all numbers $m\in\mathbb{N}^\times$ and $h\in\mathbb{C}-(U\cup\mathbb{R}_-)$ one and only

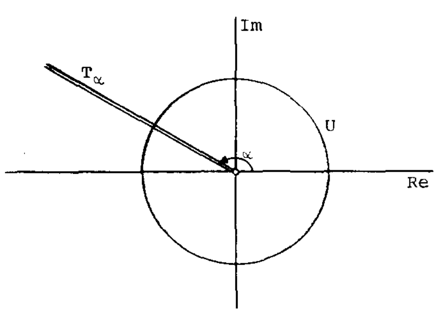

FIGURE 2

one cardinal exponential spline interpolant s_m of degree m with respect to the geometric sequence $(h^n)_{n \in \mathbb{Z}}$ that is given explicitly by (19).

The polynomials $(p_m)_{m \in \mathbb{N}^\times}$ are closely related to the exponential Euler–Frobenius polynomials (cf. Schoenberg [11, 12]). It is the aim of a forthcoming paper to establish more details of these polynomials by the integral transform method.

7. The Convergence Theorem

In view of the identities $e^{xz_k(h)} = (e^{z_k(h)})^x = h^x e^{2\pi i k x}$ $(k \in \mathbb{Z})$ that hold for all $h \in \mathbb{C} - T_\alpha$ $(\alpha \in]-\pi, +\pi])$ the pointwise convergence theorem of Schoenberg [11, 12] for the cardinal exponential spline interpolants $(s_m)_{m \in \mathbb{N}^\times}$ as their degree m tends to infinity can easily established by Theorem 5 supra. Indeed, if $r(h) = \sup(|z_0(h)/z_{-1}(h)|, |z_0(h)/z_1(h)|)$ there exists by (19) a constant $M_h > 0$ such that the estimate

$$|s_m(x) - h^x| \leqslant M_h\, |h|^x r(h)^m$$

holds for all $x \in \mathbb{R}$.

Theorem 6. *Let the weight* $h \in \mathbb{C} - (U \cup \mathbb{R}_-)$ *be given. Then the convergence property*

$$\lim_{m \to \infty} s_m(x) = h^x$$

holds for all points $x \in \mathbb{R}$.

In strong contrast to the behaviour of the cardinal exponential spline interpolants $(s_m)_{m \in \mathbb{N}^\times}$ on the line $\mathbb{R}$ the sequence $(S_m)_{m \in \mathbb{N}^\times}$ of cardinal logarithmic spline functions converges pointwise on the open right half-line $\mathbb{R}_+^\times$ only at the sequence of those points $x \in \mathbb{R}_+^\times$ where the convergence holds trivially by the interpolation property of the splines $(S_m)_{m \in \mathbb{N}^\times}$ ("Newman–Schoenberg phenomenon"; cf. the papers [5, 7, 9]). In this case a bi-infinite sequence of simple poles $\neq 0$ located on the imaginary axis of $\mathbb{C}$ is responsible for the occurrence of the pointwise convergence phenomenon.

8. Some Other Integral Representation Formulae

In [9] we have proved that the truncated power function $t \rightsquigarrow (1-t)_+^m$ admits for each exponent $m \in \mathbb{N}^\times$ the integral representation formula

$$(1-t)_+^m = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \Gamma_m(z)(mt)^{-z} \frac{dz}{i} \qquad (c > 0), \tag{23}$$

where $(\Gamma_m)_{m\in\mathbb{N}^\times}$ denotes the sequence of partial products in the classical Gauss representation of the gamma function Γ. The integral formula (23) is particularly adequate for a treatment of the pointwise convergence behaviour of the cardinal logarithmic spline functions $(S_m)_{m\in\mathbb{N}^\times}$ on the open half-line $\mathbb{R}_+^\times$ (cf. Section 7).

From our point of view the main difference between the cardinal logarithmic and the cardinal exponential case may be described roughly as follows: The identity (23) gives rise to a contour integral representation for the cardinal logarithmic splines S_m $(m\in\mathbb{N}^\times)$ that involves a line integral along the boundary of a closed vertical strip containing in its interior the imaginary axis of $\mathbb{C}$ whereas the contour of integration L_0 occurring in the integral representation formula (14) of the cardinal exponential spline interpolants s_m forms the boundary of a closed vertical strip in the open right, resp. left, half-plane of $\mathbb{C}$.

Another kind of contour integral representation of the basis splines $b_m\in\mathfrak{S}_m(\mathbb{R};\mathbb{Z})$ using rational integrands has been established by Meinardus [4]. For each point $x\in\mathbb{R}$ let γ_x denote a circuit in the complex plane $\mathbb{C}$ such that the topological index with respect to γ_x verifies the conditions

$$\begin{aligned} &\operatorname{Ind}_{\gamma_x}(n)=1 \quad \text{for} \quad [x]<n\leqslant m+1, \\ &\operatorname{Ind}_{\gamma_x}(n)=0 \quad \text{for} \quad n\leqslant [x] \quad \text{and} \quad m+1<n, \end{aligned} \qquad (n\in\mathbb{Z})$$

([ ] = Gauss symbol). Then we have the identity

$$b_m(x)=\frac{m+1}{2\pi i}\int_{\gamma_x}\frac{(z-x)^m}{\prod_{0\leqslant k\leqslant m+1}(z-k)}\,dz \qquad (x\in\mathbb{R}) \tag{24}$$

for all $m\in\mathbb{N}^\times$. The contour integral representation formula (24) allows to deduce the basic properties of the spline functions $(b_m)_{m\in\mathbb{N}^\times}$ in a simple way and may be easily extended to the case of non-equidistant bi-infinite knot sequences.

For a survey of the contour integral representations of cardinal splines and more details the reader is referred to [10].

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